ISSN: 2775-5118 VOL.5 NO.5 (2026) I.F. 9.1

ABSOLUTELY AND CONDITIONALLY CONVERGENT SERIES

Egamnazarova Jumagul Khudoyor qizi
Trainee Lecturer, Yangiyer Branch of Tashkent Institute of Chemical Technology
Yangiyer, Uzbekistan
Email: jumagul.egamnazarova1996@gmail.com
Abduraximova Dilnoza
First-year Student, Food Technology
Yangiyer Branch of Tashkent Institute of Chemical Technology

Annotation

This article discusses the concepts of absolutely and conditionally convergent series, which play an important role in mathematical analysis. The relationship between a series and the series formed by the absolute values of its terms is examined. A fundamental theorem stating that absolute convergence implies convergence is presented and justified using the Cauchy criterion. The definitions of absolute and conditional convergence are provided along with illustrative examples. In particular, the alternating harmonic series is analyzed as a classical example of a conditionally convergent series, and its connection to the Maclaurin expansion of the logarithmic function is demonstrated. The results highlight the differences between absolute and conditional convergence and their significance in the study of infinite series.

Keywords: Absolutely convergent series, conditionally convergent series, infinite series, Cauchy criterion, convergence, divergence, alternating series, harmonic series, Maclaurin series, logarithmic function.

Introduction

Let us assume that the following series is given:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

The terms of this series are real numbers with arbitrary signs. (Such a series is usually called a series with arbitrary terms.)

From the absolute values of the terms of series (1), we construct the series:

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_n| + \ldots + |a_n| + \ldots \qquad (2)$$

Theorem 1

If series (2) is convergent, then series (1) is also convergent.

Proof:

Assume that series (2) is convergent.

Then, according to the Cauchy criterion for convergence of series, for any ,

$\forall \varepsilon > 0, \quad \exists n_0 \in N, \quad \forall n > n_0 \quad m = 1,2,3\ldots$

$$|a_{n+1}| + |a_{n+2}| + \ldots + |a_{n+m}| < \varepsilon$$

It is clear that:

$$|a_{n+1} + a_{n+2} + \ldots + a_{n+m}| \leq |a_{n+1}| + |a_{n+2}| + \ldots . + |a_{n+m}|.$$

From these relations, it follows that:

$\forall \varepsilon > 0, \quad \exists n_0 \in N, \forall n > n_0, \quad m = 1,2,3,\ldots$ da $|a_{n+1} + a_{n+2} + \ldots + a_{n+m}| < \varepsilon$

Hence, by the Cauchy criterion, series (1) is convergent.►

Definition 1

If the series

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent, then the series,

$$\sum_{n=1}^{\infty} a_n$$

is called an **absolutely convergent series**.

Example:

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}(-1)^{n-1} = 1 - \frac{1}{2^\alpha} + \frac{1}{3^\alpha} - \frac{1}{4^\alpha} + \ldots + \frac{(-1)^{n-1}}{n^\alpha} + \ldots$$

$\alpha > 1$ is absolutely convergent, because

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^\alpha}(-1)^{n-1} \right| = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} - \frac{1}{4^\alpha} + \ldots + \frac{1}{n^\alpha} + \ldots$$

is a p-series (generalized harmonic series) that converges when $\alpha > 1$.

Definition 2

If the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent, but the series,

$$\sum_{n=1}^{\infty} |a_n|$$

is divergent, then the series

$$\sum_{n=1}^{\infty} a_n$$

is called a **conditionally convergent series.**

Example:

Consider the series:

$$\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n-1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n-1}}{n} + \ldots$$

This is a conditionally convergent series.

Let S_n denote its partial sum:

$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n-1}}{n} \quad (3)$$

It is known that the Maclaurin series expansion of the function $\ln(1+x)$ is:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}x^n}{n} + R_{n+1}(x), \quad 0 \le x \le 1$$

$$|R_{n+1}(x)| < \frac{1}{n+1}$$

In particular, for x=1:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n-1}}{n} + R_{n+1}(1)$$

$$|R_{n+1}(1)| < \frac{1}{n+1} \quad (4)$$

From relations (3) and (4), it follows that:

$$\ln 2 = S_n + R_{n+1}(1)$$

$$|S_n - \ln 2| < \frac{1}{n+1}$$

as $n \to \infty$. Hence, the given series is convergent.

At the same time, the series formed by the absolute values of its terms:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

is the harmonic series, which is known to be divergent.

Therefore, the given series is **conditionally convergent**.

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